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PART II

Theory of Production and Cost

In older textbooks it was conventional to define production as "the creation of utility" where utility meant "the ability of a good or service to satisfy a human want." In one respect this definition is too broad to have much specific content. On the other hand, however, it definitely points out that "production" embraces a wide range of activities and not *only* the fabrication of material goods. Rendering legal advice, writing a book, showing a motion picture, servicing a bank account are all examples of "production." It is rather difficult to specify the inputs used in producing the outputs of these illustrative cases. Nevertheless, most people would probably agree that some kinds of technical and intellectual skills are required to perform the services.

Thus while "production" in a general sense refers to the creation of any *good or service* people will buy, the concept of production is much clearer when we speak only of *goods*. In this case it is simpler to specify the precise inputs and to identify the quantity and quality of output. Producing a bushel of wheat requires, in addition to suitable temperature and rainfall, a certain amount of arable land, seed, fertilizer, the services of agricultural equipment such as plows and combines, and human labor.

Even in our presently advanced state of automation, every act of production requires the input of human resources. Other inputs are usually required as well. In particular, production normally requires various types of capital equipment (machines, tools, conveyors, buildings) and raw or processed materials. The theory of production consists of an analysis of *how* the businessman—given the "state of the art" or technology—combines various inputs to produce a stipulated output in an economically efficient manner.

Since the concept of production is clearer when applied to goods rather than services, our discussion will be restricted to production in agricultural and manufacturing industries. The student should be aware, nevertheless, that problems of resource allocation in service trades and industries are not less serious because they are less discussed in this text. Indeed, as the population becomes more and more concentrated in the under-twenty and over-sixty-five age groups, the importance of services relative to goods increases. The principles of production studied here are as applicable to the output of services as to the output of goods, even though the application may be more difficult in the former case.

The same statement applies to the theory of cost. It is simpler to study a manufacturing business engaged in producing a specific good. Even then both costing and pricing are difficult matters—but not nearly so difficult as in service trades and government. Thus our discussion is restricted to producers of goods.

The theory of cost consists of an analysis of the costs of production—how costs are determined from a knowledge of the production function, the effects of diminishing returns, cost in the short and long runs, the "four cost curves," and so on. But more importantly, it establishes the basis for studying the pricing practices of business firms, which occupies Part III.

Chapter 6 PRODUCTION WITH ONE VARIABLE INPUT

6.1 INTRODUCTION

Production processes typically require a wide variety of inputs. These are not as simple as "labor," "capital," and "materials;" many qualitatively different types of each are normally used to produce an output. To clarify the analysis, this chapter introduces some simplifying assumptions whose purpose is to cut through the complexities of dealing with hundreds of different inputs. Thus our attention can be focused upon the essential principles of production.

More specifically, we assume there is only one *variable input.* In subsequent discussion, this variable input is usually called "labor," although any other input could just as well be used. Second, we assume this variable input can be combined in different proportions with one *fixed input* to produce various quantities of output. The fixed input is called "land;" our discussion is thus principally concerned with one specific example of production: agricultural output.

Finally, note that three assumptions are actually embodied in the two propositions stated above: (a) there is only one variable input; (b) there is only one fixed input; and (c) inputs may be combined in various proportions to produce the commodity in question.

6.1.a—Fixed and Variable Inputs, the Short and Long Runs

In analyzing the process of physical production and the closely related costs of production, it is convenient to introduce an analytical fiction: the classification of inputs as fixed and variable. Accordingly, a *fixed input* is defined as one whose quantity cannot readily be changed when market conditions indicate that an immediate change in output is desirable. To be sure, no input is ever *absolutely* fixed, no matter how short the period of time under consideration. But frequently, for the sake of analytical simplicity, we hold some inputs fixed, reasoning perhaps that while these inputs are in fact variable the cost of immediate variation is so great as to take them out of the range of relevance for the particular decision at hand. Buildings, major pieces of machinery, and managerial personnel are examples of inputs that cannot be rapidly augmented or diminished. A *variable input,* on the other hand, is one whose quantity may be changed almost instantaneously in response to desired changes in output. Many types of labor services and the inputs of raw and processed materials fall in this category.

Corresponding to the fiction of fixed and variable inputs, economists introduce another fiction, the short and long runs. The *short run* refers to that period of time in which the input of one or more productive agents is fixed. Therefore, changes in output must be accomplished exclusively by changes in the usage of variable inputs. Thus if a producer wishes to expand output in the short run, he must usually do so by using more hours of labor service with the existing plant and equipment. Similarly, if he wishes to reduce output in the short run, he may discharge certain types of workers; but he cannot immediately "discharge" a building or a diesel locomotive, even though its usage may drop to zero.

In the long run, however, even this is possible, for the *long run* is defined as that period of time (or planning horizon) in which all inputs are variable. The long run, in other words, refers to that time in the future when output changes can be achieved in the manner most advantageous to the businessman. For example, in the short run a producer may be able to expand output only by operating his existing plant for more hours per day. This, of course, entails paying overtime rates to workers. In the long run, it may be more economical for him to install additional productive facilities and return to the normal work day.

In this part of the book we are mostly concerned with the short-run theory of production, combining different quantities of variable inputs with a specific quantity of fixed input to produce various quantities of output. The long-run organization of production is largely determined by the relative cost of producing a desired output by different input combinations. Discussion of the long run is thus postponed until Chapters 7 and 8.

6.1.b—Fixed or Variable Proportions

As already indicated, our discussion focuses largely upon the use of a *fixed* amount of one input and a *variable* amount of another to produce *variable* quantities of output. This means our attention is restricted mainly to production under conditions of *variable proportions.* The *ratio of input quantities* may vary; the businessman,

therefore, must determine not only the level of output he wishes to produce but also the optimal proportion in which to combine inputs (in the long run).

There are two different ways of stating the principle of variable proportions. First, variable-proportions production implies that output can be changed in the short run by changing the amount of variable inputs used in cooperation with the fixed inputs. Naturally, as the amount of one input is changed, the other remaining constant, the *ratio* changes. Second, when production is subject to variable proportions, the *same* output can be produced by various combinations of inputs—that is, by different input ratios. This may apply only to the long run, but it is relevant to the short run when there is more than one variable input.

Most economists regard production under conditions of variable proportions as typical of both the short and long run. There is certainly no doubt that proportions are variable in the long run. When making an investment decision a businessman may choose among a wide variety of different production processes. As polar opposites, an automobile can be almost handmade or it can be made by assembly-line techniques. In the short run, however, there may be some cases in which output is subject to fixed proportions.

Fixed-proportions production means there is one, and only one, ratio of inputs that can be used to produce a good. If output is expanded or contracted all inputs must be expanded or contracted so as to maintain the fixed input ratio. At first this might seem the usual condition: one man and one shovel produce a ditch, two parts hydrogen and one part oxygen produce water. Adding a second shovel or a second part of oxygen will not augment the rate of production.

But in actuality examples of fixed-proportions production are hard to come by. Even the production of chemical compounds is subject to variable proportions! It is true that hydrogen and oxygen must be used in the fixed ratio of 2:1 to produce water. But the *yield* of water for any given amount of hydrogen and oxygen depends upon the amount of catalyst used. Proportions are indeed variable from the standpoint of the catalyst, not only in this one instance but in the production of almost every chemical compound.

The hydrogen-oxygen-water illustration serves as a convenient introduction to a general view of production processes. One might say that, in the short run, there are three classes of productive inputs. First, there are certain fixed inputs whose quantity cannot be varied in the short run. Second, there are variable inputs whose usage may be changed. Finally, there are "ingredient" inputs whose quantities may be readily changed but must bear fixed proportions one to the other and to output.

It is not difficult to find examples of "ingredient" inputs. Each brand of cigarettes contains its own special blend of tobaccos. That is, various tobaccos are blended in fixed proportions. And a fixed amount of tobacco blend must be used in each cigarette produced. But the production of cigarettes requires more than the fixed-proportions ingredient inputs. Certain capital equipment—rolling machines, packaging machines, and the like—must be used and human labor services are necessary. In the short run, the building and capital equipment are fixed inputs and most labor services are variable.

In the discussion of production the fixed and variable inputs are stressed. Ingredient inputs are necessary; and they must be used in fixed or relatively fixed proportions or else the quality or character of the output will change. The businessman has little or no choice in this regard. Hence our attention is directed to those aspects of production over which a businessman can exert control.

6.2 THE PRODUCTION FUNCTION

The discussion so far, especially in subsection 6.1.b, has emphasized that the quantity of output depends upon, or is a function of, the quantities of the various inputs used. This relationship is more formally described by a *production function* associating physical output with input.

Definition: a production function is a schedule (or table, or mathematical equation) showing the maximum amount of output that can be produced from any specified set of inputs, given the existing technology or "state of the art." In short, the production function is a catalogue of output possibilities.

6.2.a—Total Output or Product

The production function may be shown as a table or, alternatively, as a mathematical equation. In either case the short-run production function gives the total (maximum) output obtainable from different amounts of the variable input, given a specified amount of the fixed input and the required amounts of the ingredient inputs.[1]

[1] Let q represent the quantity of output. Suppose there is one variable input x, one fixed input y, and one required ingredient input w. Mathematically, the short-run production function could be written as $q = f(x|y, w)$, where the vertical bar means "given."

As an example, consider an experiment in the production of wheat on three acres of land. The fixed input is land, the ingredient input is seed; and the variable input is man-years of labor time. The output is

TABLE 6.2.1

OUTPUT OF WHEAT IN BUSHELS ON THREE-ACRE TRACTS OF LAND

Tract No.	*Number of Workers*	*Total Output*
1	1	10
2	2	24
3	3	39
4	4	52
5	5	61
6	6	64
7	7	65
8	8	64

bushels of wheat. An agricultural experiment station blocks off eight tracts of land, each containing three acres. The first tract is worked for a producing season by one man; the second tract is worked by two men; and so on until the eighth tract is worked by eight men. Total output on the various tracts of land might be as shown in Table 6.2.1.

The hypothetical data in Table 6.2.1 are graphed in Figure 6.2.1.

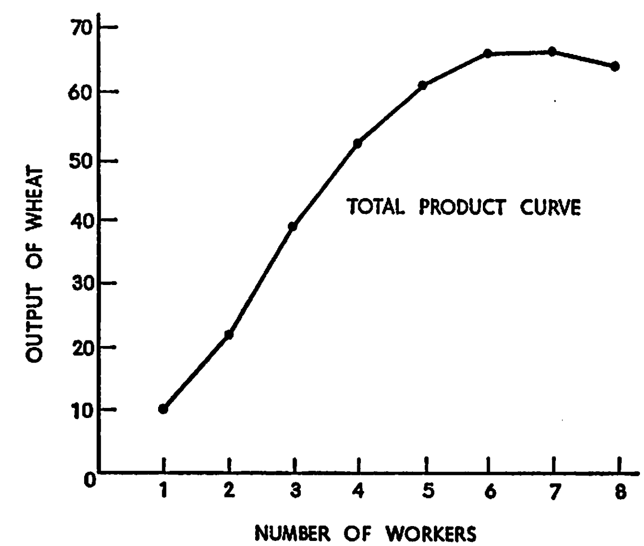

FIGURE 6.2.1

TOTAL PRODUCT CURVE OBTAINED FROM HYPOTHETICAL DATA IN TABLE 6.2.1

Since output is a function of input, the former—output or total product—is plotted on the vertical axis. The independent variable—number of workers—is plotted on the horizontal axis. Joining the successive points by straight-line segments, one obtains the total product curve. It is important to note that the curve first rises slowly, then more rapidly, and then more slowly again until it finally reaches a maximum and begins to decrease. The reasons for this curvature lie in the familiar principle of diminishing marginal physical returns.

6.2.b—Average and Marginal Products

Two important relationships between inputs affect the level of output and the relationship between output and input. The first of these is the *ratio* in which the inputs are used (in the present illustration, the land-labor ratio). Second, for any given input ratio the *scale* of inputs, or the absolute magnitude of input quantities, is important. To analyze scale effects there must be two or more variable inputs; so our attention in this chapter is confined to the effects incident to changes in the input ratio.

Table 6.2.2 is an expanded version of Table 6.2.1, with some change in the *Total Output* column. The first two columns still indicate the tract number and the number of workers on each tract. The third column shows the input ratio for each tract, or the average number of acres of land per worker. The fourth column reports the total output for each tract of land, while the fifth column shows the average output per worker, or the output-labor ratio. Finally, column six contains the entries for marginal product.

Definition: the average product of an input is total product divided by the amount of the input used to produce this output. Thus average product is the output-input ratio for each level of output and the corresponding volume of input.

Definition: the marginal product of an input is the addition to total product attributable to the addition of one unit of the variable input to the production process, the fixed input remaining unchanged. (*Note:* marginal product refers only to comparisons of the results of simultaneous experiments and not to the successive addition of units of the variable input in one experiment.[2])

[2] Consider the production function in footnote 1: $q = f(x|y, w)$. The average product of the variable input x is

$$\frac{q}{x} = \frac{f(x|y, w)}{x},$$

and the marginal product is

$$\frac{dq}{dx} = \frac{df(x|y, w)}{dx}.$$

TABLE 6.2.2

AVERAGE AND MARGINAL PRODUCTS AND THE INPUT RATIO FOR THREE-ACRE TRACTS

Tract Number	Number of Workers	Land-Labor Ratio	Total Output	Average Product of Labor	Marginal Product of Labor
1	1	3.0	10	10	—
2	2	1.5	24	12	14
3	3	1.0	39	13	15
4	4	.75	52	13	13
5	5	.60	61	12.2	9
6	6	.50	66	11.0	5
7	7	.43	66	9.4	0
8	8	.37	64	8.0	−2

Table 6.2.2 and its accompanying graph, Figure 6.2.2, illustrate several important features of a typical production process. First, both average and marginal products first rise, reach a maximum, and then decline. In the limit, average product could decline to zero because total

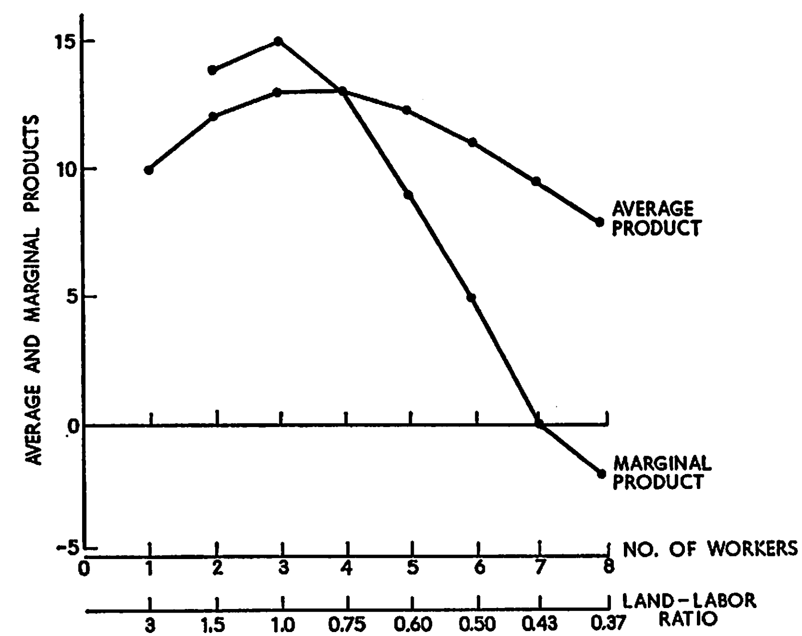

FIGURE 6.2.2

AVERAGE AND MARGINAL PRODUCTS OBTAINED FROM DATA IN TABLE 6.2.2

product itself could conceivably decline to this point. Marginal product, on the other hand, may actually become negative—indeed, many economists suggest that the marginal product of agricultural workers in some underdeveloped countries is in fact negative. In the present example, the marginal product of labor becomes negative because the variable input is used too intensively with the fixed input (land).

A second feature of significance is that marginal product exceeds average product when the latter is rising, equals average product when the latter is a maximum, and lies below average product when the latter is falling. This proposition follows readily from the definitions of marginal and average product. So long as the *addition* to a total is greater than the previous average, the average must increase. If the *addition* to the total is less than the previous average, the newly computed average must be less. Thus, since the *additions* first rise and then decline so also must the averages; and the two curves must intersect at the point where the average curve reaches its maximum.[3]

The third feature to note is that as the input (land-labor) ratio declines, the output-labor ratio first rises and then declines indefinitely. The marginal product of labor behaves in a similar manner, as will be explained below.

6.2.c—Law of Diminishing Marginal Physical Returns

The shape of the marginal product curve in Figure 6.2.2 graphically illustrates an important principle that is already familiar to you: the "law" of diminishing marginal physical returns.

When the outputs of tracts one and two are compared (Table 6.2.2), one sees that using two workers rather than one increases output by fourteen bushels, the marginal product of labor when there are two workers. Similarly, comparing tracts two and three, the use of a third worker augments output by fifteen bushels. The marginal physical product of labor increases as the number of workers increases. This may well happen when the land-labor ratio is very high.

[3] For simplicity, ignore the fixed and ingredient inputs and write the production function as $q = f(x)$. Thus average product is q/x and marginal product is $\frac{dq}{dx}$. Average product is a maximum when

$$\frac{d\left(\frac{q}{x}\right)}{dx} = \frac{1}{x}\left[\frac{dq}{dx} - \frac{q}{x}\right] = 0 \cdot$$

Since $x > 0$, $\frac{1}{x}\left[\frac{dq}{dx} - \frac{q}{x}\right]$ can equal zero only when $\frac{q}{x} = \frac{dq}{dx}$.

Thus marginal product equals average product when the latter is at its maximum value.

Ultimately, however, as the input ratio declines so also must the marginal product of the variable input. When the number of units of the variable input increases, each unit, so to speak, has on the average fewer units of the fixed input with which to work. At first, when the fixed input is relatively plentiful, more intensive utilization of fixed inputs by variable inputs may increase the marginal output of the variable input. Nonetheless, a point is quickly reached beyond which an increase in the intensity of use of the fixed input yields progressively less and less additional returns. Psychologists have even found that this holds true for consecutive study time.

Principle (the law of diminishing marginal physical returns): as the amount of a variable input is increased, the amount of other (fixed) inputs held constant, a point is reached beyond which marginal product declines.

6.2.d—Product Curves for Different Amounts of the Fixed Input

The fixed input is a parameter that causes the whole set of product curves to shift. Generally, the greater the amount of fixed input available the greater the input ratio and the greater also the total, average, and marginal products. Increasing the land-labor ratio increases the amount of fixed input available per unit of variable input. This normally results in an increase in the marginal, and hence in the average and total, product of the variable input.

This proposition is illustrated in Table 6.2.3 and shown graphically in Figures 6.2.3 and 6.2.4.[4] The first row in each part of the table

TABLE 6.2.3

TOTAL, AVERAGE, AND MARGINAL PRODUCTS ON ONE- AND TWO-ACRE TRACTS OF LAND

Product	*Size of Tract*	*Number of Workers* 1	2	3	4	5	6
Total	1 Acre	20	30	37.5	44	50	55.5
	2 Acres	25	40	52.5	64	75	85.5
Average	1 Acre	20	15	12.5	11	10	9.25
	2 Acres	25	20	17.5	16	15	14.25
Marginal	1 Acre	—	10	7.5	6.5	6	5.5
	2 Acres	—	15	12.5	11.5	11	10.5

[4] As you will see in section 6.3 below, when the average product of the variable input is increasing the marginal product of the fixed input is negative. Hence the results of this subsection hold only for the range of production from maximum average product to zero marginal product of the variable input.

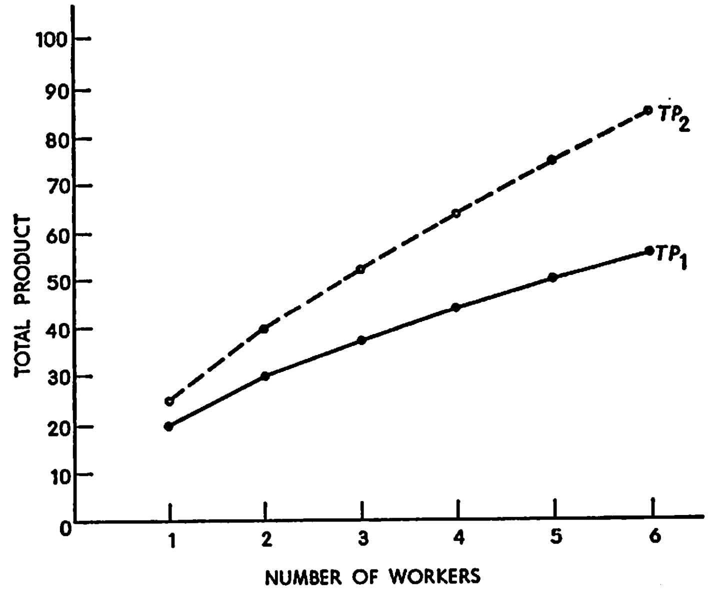

FIGURE 6.2.3

TOTAL PRODUCT CURVES FOR ONE- AND TWO-ACRE TRACTS OF LAND

shows data from an hypothetical agricultural experiment on one-acre tracts of land. The second row shows the corresponding data for experiments on two-acre tracts. Comparing each set of rows, the table shows that for each amount of variable input, the total, average, and marginal products are greater on the two-acre tracts of land. The table, together with the associated figures, clearly shows that over the relevant range of production (see section 6.3 below) augmenting the fixed input augments the productivity of the variable input.

6.3 THE STAGES OF PRODUCTION

This study of production has so far focused attention upon one specific, discrete production function given in tabular form. We turn now to a more general formulation in which both discrete and continuous production functions are used.

6.3.a—Geometry of Average Product Curves

A typical form of the (continuous) total product curve is shown in Figure 6.3.1. In this, as in all other one-variable-input product graphs, units of the variable input are plotted on the abscissa and total product is plotted on the ordinate.

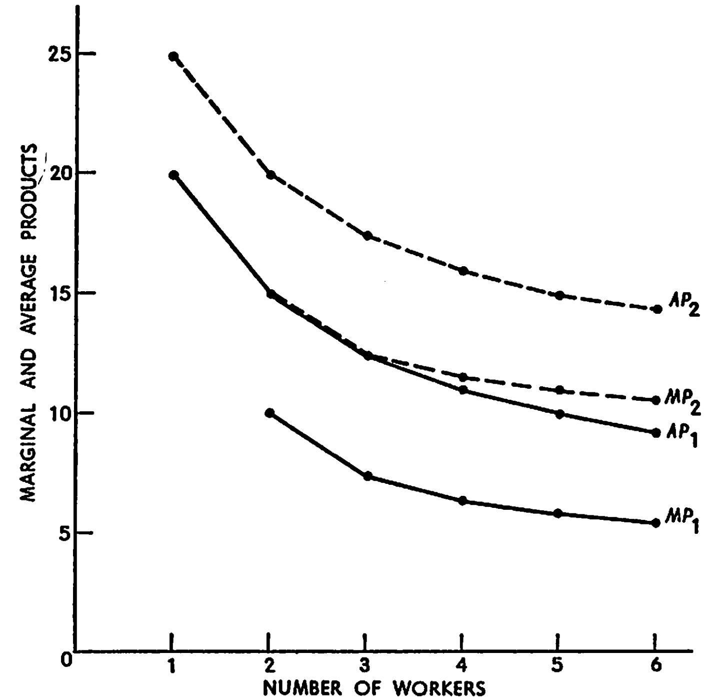

FIGURE 6.2.4

AVERAGE AND MARGINAL PRODUCT CURVES FOR ONE- AND TWO-ACRE TRACTS OF LAND

Given the total product curve TP, we wish to find average product. First, from its definition average product is total product divided by the number of units of the variable input used to produce it, or the output-variable input ratio. Producing total output $OR = DA$ requires OD

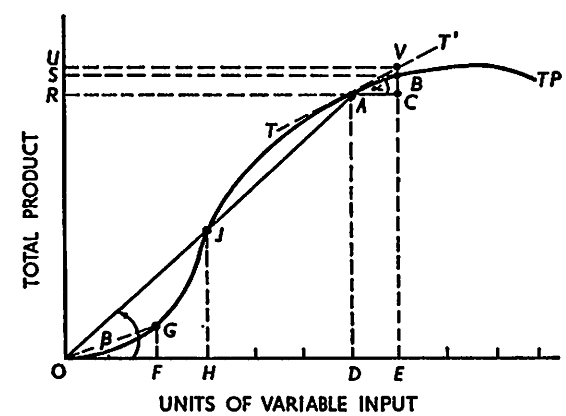

FIGURE 6.3.1

GEOMETRY OF MARGINAL AND AVERAGE PRODUCT CURVES units of the variable input. Thus the average product of OD units of variable input is DA/OD. Similarly, the average product of OF units of variable input is FG/OF and of OH units is HJ/OH. In each case, to obtain the average product corresponding to a given point on the total product curve, we found the slope of the line joining the origin with the point in question. In other words, we found the tangent of the angle formed by the abscissa and the line from the origin to the given point on the total product curve.

As we have seen, the average product corresponding to point A is DA/OD, but this is precisely the slope of the line OA, or the tangent of the angle β. Notice also that average product must be the same for OH as for OD units of the variable input because the slopes of OJ and of OA are identical (in each case, average product is the tangent of angle β). Since average product is rising for movements along TP from the origin to point J, and since it is obviously falling for movements from A to B, there is reason to suspect that average product reaches its maximum at a point between J and A on the total product curve.

Average product does, in fact, attain its maximum at an intermediate point, as may be seen more clearly in Figure 6.3.2. Points Q and R in Figure 6.3.2 correspond to points J and A, respectively, in Figure 6.3.1, in that each pair of points lies on a common ray from the origin.

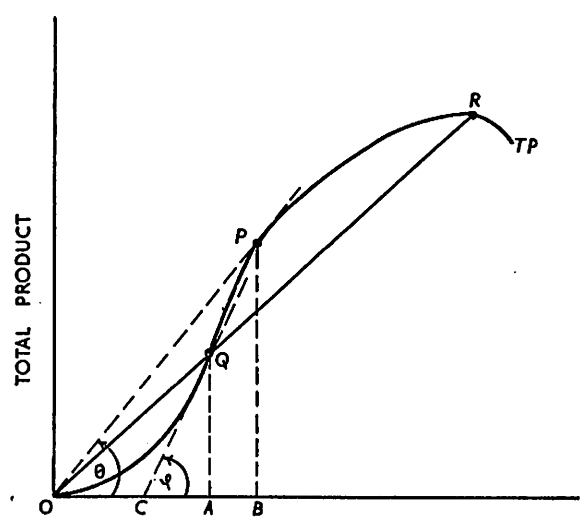

FIGURE 6.3.2

MAXIMA OF AVERAGE AND MARGINAL PRODUCTS

Thus the average product at point Q is equal to the average product at point R. Since average product is the slope of a ray from the origin to a point on the curve, average product is a maximum when the slope of the line is steepest. This occurs, of course, when the line from the origin is just tangent to the total product curve, at point P with angle θ in Figure 6.3.2.

As one moves from the origin through point Q toward point P, the line from the origin to the curve becomes steeper. Similarly, as one moves from P toward point R, the line moves downward, becoming less steep. Thus we have proved the following important points.

Relationships: average product corresponding to any point on the total product curve is given by the slope of a ray from the origin to the point in question. Average product attains its maximum value when this line is just tangent to the total product curve.

6.3.b—Geometry of Marginal Product Curves

Using Figures 6.3.1 and 6.3.2 again, similar qualitative and quantitative relationships can be found for the marginal product curve.

Turn first to Figure 6.3.1. By definition, marginal product is the addition to total product attributable to the addition of one unit (or a small amount) of the variable input to a given amount of the fixed input. Let the amount of variable input increase from OD to OE, or by the amount $DE = AC$. Output consequently increases from OR to OS, or by the amount $RS = BC$. Marginal product is, therefore, BC/AC. In this discrete case, there is no convenient slope measurement because the arc AB is not linear. That is, a unique slope measure cannot be obtained because the slope of the angle formed by arc AB and line AC changes over the interval $DE = AC$.

But let us suppose for a minute that the total product curve were linear from A to the point V. Then an increment of amount DE in the variable input would cause output to increase from OR to OU, or by $RU = CV$. In this case, marginal product would be CV/AC, or the slope of angle α. The measure CV/AC overstates the true magnitude of marginal product, BC/AC. However, as the increment of variable input becomes smaller and smaller the approximation becomes better and better. In the limit, for a very tiny increase in variable input the slope of the tangent to point A, labeled TT', approaches the true slope of the total product curve. Hence for sufficiently small changes in the variable input, the slope of the total product curve at any point is a good approximation of marginal product.[5]

The slope of a curve at any point is given by the slope of its tangent at that point. Thus the marginal product corresponding to point Q in Figure 6.3.2 is the slope of the line CQ, or the tangent of angle $\phi = AQ/CA$. As Figure 6.3.2 is constructed, marginal product is a maximum when OA units of variable input are used. This is true because the slope of the tangent to the total product curve is steeper at point Q than at any other point.

Other interesting relationships can be determined from Figure 6.3.2. First, recall that maximum average product is associated with OB units of variable input and corresponds to point P. Hence marginal product attains its maximum at a lower level of variable input than does average product. Second, notice that the tangent to the total product curve at point P—the line whose slope gives marginal product corresponding to point P—is the line OP. We have already seen in subsection 6.3.a that the slope of OP also gives average product associated with point P and that average product attains its maximum value at that point. Hence, as we have seen previously, marginal product equals average product when the latter is at its maximum.

The principal information contained in this subsection can be summarized as follows.

Relationships: marginal product corresponding to any point on the total product curve is given by the slope of the tangent to the curve at that point. Marginal product attains its maximum value when the slope of the tangent is steepest. The point of maximum marginal product occurs at a smaller level of variable input than does maximum average product; and marginal product equals average product when the latter attains its maximum value.

6.3.c—Total, Average, and Marginal Products

The relationships discussed in the two preceding subsections are illustrated in Figure 6.3.3. In this graph one can see not only the

[5] Let $q = f(x)$ be the production function. If the increment of variable input is denoted Δx, the new output is $f(x + \Delta x)$. Thus, by definition, marginal product is

$$MP_x = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

But also by definition, the derivative of $f(x)$ is

$$\frac{dq}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Hence in the limit, marginal product *is* the slope (dq/dx) of the total product curve. For finite changes, the slope is an approximation of marginal product.

relationship between marginal and average products but also the relationship of these two curves to total product.

Consider first the total product curve. For very small amounts of the variable input, total product rises gradually. But even at a low level of input it begins to rise quite rapidly, reaching its maximum slope (or rate of increase) at point one. Since the slope of the total product curve equals marginal product, the maximum slope (point one) must correspond to the maximum point on the marginal product curve (point four).

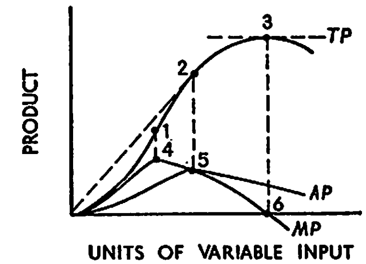

FIGURE 6.3.3

TOTAL, AVERAGE, AND MARGINAL PRODUCTS

After attaining its maximum slope at point one, the total product curve continues to rise. But output increases at a decreasing rate, so the slope is less steep. Moving outward along the curve from point one, soon the point is reached at which a ray from the origin is just tangent to the curve (point two). Since tangency of the ray to the curve defines the condition for maximum average product, point two lies directly above point five.

As the quantity of variable input is expanded from its value at point two, total product continues to increase. But its rate of increase is progressively slower until point three is finally reached. At this position total product is at a maximum; thereafter it declines until it (conceivably) reaches zero again. Over a tiny range around point three, additional input does not change total output. The slope of the total product curve is zero. Thus marginal product must also be zero. This is shown by the fact that points three and six occur at precisely the same input value. And since total product declines beyond point three, marginal product becomes negative.

Most of the important relationships have so far been discussed with reference to the total product curve. To emphasize certain relationships, however, consider the marginal and average product curves in Figure 6.3.3. Marginal product at first increases, reaches a maximum at point four (the point of diminishing marginal physical returns), and declines thereafter. It eventually becomes negative beyond point six, at which total product is at its maximum.

Average product also rises at first until it reaches its maximum at point five, where marginal and average products are equal. It subsequently declines, conceivably becoming zero when total product itself

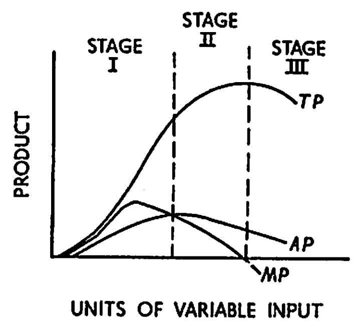

FIGURE 6.3.4

STAGES OF PRODUCTION

becomes zero. Finally, one may observe that marginal product exceeds average product when the latter is increasing and is less than average product when the latter is decreasing.

6.3.d—Three Stages of Production

The relationships among total, average, and marginal products are used to define three stages of production, as illustrated in Figure 6.3.4.

Stage I covers that range of variable input over which average product is rising. In other words, stage I corresponds to increasing *average* returns of the variable input. But, as explained in the next subsection, increasing average returns to the variable input are associated with negative marginal returns to the fixed input. The fixed input is present in uneconomically large proportion to the variable input in stage I. A rational producer would never operate in this range of production. If market conditions dictated such a small level of total output, the output would be produced by using fewer units of the fixed input—the entire set of product curves would be *shifted* from the position shown in Figure 6.3.4.

Production would also never occur in stage III, as is more or less

obvious from the graph. Stage III is defined as the range of negative marginal product or declining total product. Additional units of variable input during this stage of production actually cause a decrease in total output. Even if units of variable input were *free,* a rational producer would not employ them beyond the point of zero marginal product because their use entails a reduction in total output. If market conditions dictate an expansion of output, additional units of the fixed input must be used, shifting the entire set of product curves upward (as illustrated in Figures 6.2.3 and 6.2.4).

In stage III the variable input is combined with the fixed input in uneconomically large proportions. In terms of agriculture, land is cultivated too intensively. Indeed, the point of zero marginal product of the variable input is called the *intensive margin.* Similarly, in suggestive terminology, at the point of maximum average product of the variable input the cultivation of land is extensive; and the point of maximum average product of the variable input is called the *extensive margin.*[6]

We have now eliminated stages I and III. Production must occur in stage II—between the extensive margin and the intensive margin, or over the range of variable input from maximum average product to zero marginal product. If output must be produced in quantities not covered by production in stage II, there must be some change in the quantity of fixed input. If a smaller output is desired, the units of fixed input must be reduced; if a larger output is required, it can be achieved only by augmenting fixed input.

6.3.e—Symmetry of the Stages of Production

The purpose of this concluding subsection is to demonstrate that the stages of production are symmetrical when viewed from the standpoints of the variable and of the fixed input. This, in turn, will explain the so-far ambiguous statement that rising average returns to the variable input correspond to negative marginal returns to the fixed input.

Table 6.3.1 reproduces the example discussed in section 6.2. Eight three-acre tracts of land are simultaneously cultivated. One worker is on the first tract; two workers are on the second; and so on until eight workers cultivate the eighth tract. Table 6.3.1 shows the total, average,

[6] As shown in subsection 6.3.e, the point of maximum average product of the variable input corresponds to the point of zero marginal product of the fixed input. Hence the extensive margin, from the standpoint of the variable input, is the intensive margin in terms of the fixed input, and vice versa.

and marginal products of labor (the variable input) determined from this experiment.

We can use Table 6.3.1 to derive the implicit total and marginal products of land, the fixed input. From the last row in Table 6.3.1 it is seen that three acres of land cultivated by eight workers produces sixty-four units of output. In this experiment, each worker on the average cultivates three-eighths acres of land. Hence, if one worker were on

TABLE 6.3.1

TOTAL, AVERAGE, AND MARGINAL PRODUCTS OF LABOR IN CULTIVATING THREE-ACRE TRACTS OF LAND

Acres of Land	*Number of Workers*	*Land-Labor Ratio*	*Total Product (Labor)*	*Average Product (Labor)*	*Marginal Product (Labor)*
3	1	3/1	10	10	—
3	2	3/2	24	12	14
3	3	3/3	39	13	15
3	4	3/4	52	13	13
3	5	3/5	61	12.2	9
3	6	3/6	66	11	5
3	7	3/7	66	9.4	0
3	8	3/8	64	8	−2

three-eighths acres of land, total product would be one-eighth as great as when eight workers are on three acres.[7] Therefore, total product with three-eighths acres and one worker is one-eighth of sixty-four, or eight.

Similarly, from the next-to-last row of Table 6.3.1, three acres of land worked by seven units of labor produce sixty-six units of output. Therefore, three-sevenths acres of land cultivated by one worker will produce one seventh of this output, or one seventh of sixty-six, which equals nine and four-tenths. Proceeding in this manner from the botton to the top of Table 6.3.1, we can determine the total product of land, as shown in the fourth column of Table 6.3.2. One should notice that the

[7] This statement is strictly true only when production is subject to constant returns to scale. "Constant returns to scale" means that if the input of *each* productive agent is multiplied by a number m (however large or small), output is also multiplied by m. In other words, "constant returns to scale" implies that two workers, each cultivating one acre of land, can in total produce exactly the same output as two workers jointly cultivating two acres of land.

Mathematically, the production function must be homogeneous of degree one if constant returns to scale obtain. By the definition of first-degree homogeneity, if $q = f(x, y)$, then $f(mx, my) = mf(x, y) = mq$. This is precisely what is meant by constant returns to scale.

method of computation necessarily implies that for each land-labor ratio, the total product of land is precisely the same as the average product of labor.

Next we need to determine the marginal product per acre of land. In Table 6.3.2, when the amount of land changes from three-eighths acres to three-sevenths acres, the total product of land increases from eight to nine and four-tenths, or by one and four-tenths units. The fixed input has increased by three-sevenths minus three-eighths, or three-fifty-

TABLE 6.3.2

TOTAL AND MARGINAL PRODUCTS OF LAND WHEN VARIOUS TRACTS ARE CULTIVATED BY ONE WORKER

Acres of Land	*Number of Workers*	*Land-Labor Ratio*	*Total Product (Land)*	*Average Product (per acre of land)*	*Marginal Product (per acre of land)*
3/8	1	3/8	8	21.3	—
3/7	1	3/7	9.4	22.6	26.1
3/6	1	3/6	11	22	22.4
3/5	1	3/5	12.2	20.8	12
3/4	1	3/4	13	17.3	5.3
3/3	1	3/3	13	13	0
3/2	1	3/2	12	8	−2
3/1	1	3/1	10	3.3	−1.3

sixths. Hence, the marginal product of an additional acre at this level of fixed input is $1.4\ (^{56}/_{3}) = 26.1$. Similarly, when the units of fixed input increase from three-sevenths acres to three-sixths acres, total product increases from nine and four-tenths to eleven, or by one and six-tenths. Hence the marginal product is $1.6\ (^{42}/_{3}) = 22.4$. Proceeding in this manner, the marginal product of land can be determined for the various acreages in the experiment. The figures are shown in the last column of Table 6.3.2.

Finally, the average product per acre of land may be derived in an analagous fashion. In combination with one worker, three-eighths acres of land can produce eight units of output. Hence one acre of land has an average product of $8\ (^{8}/_{3}) = 21.3$. Similarly, three-sevenths acres of land produce nine and four-tenths units of output. Hence at this input point, the average product per acre is $9.4\ (^{7}/_{3}) = 22.6$. By employing this method for each value of land input, the average product per acre of land is obtained (next-to-last column, Table 6.3.2).

The hypothetical data in Tables 6.3.1 and 6.3.2 are plotted in

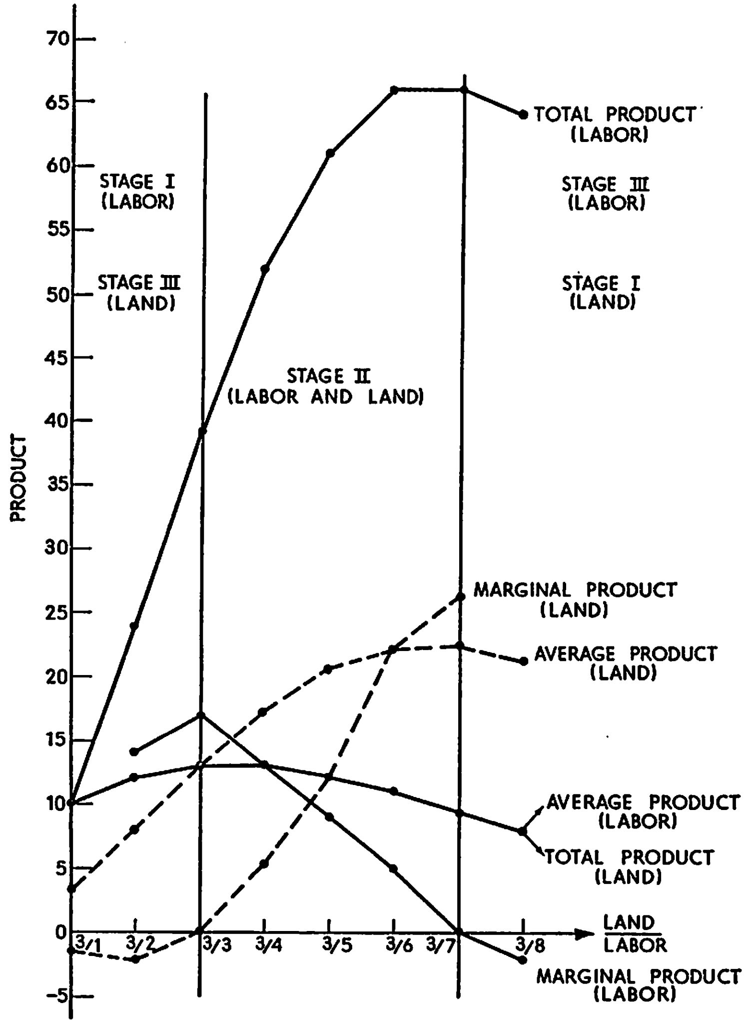

FIGURE 6.3.5

PRODUCT CURVES FOR LABOR AND LAND FOR COMMON LAND-LABOR RATIOS

Figure 6.3.5. Notice carefully that the horizontal axis represents the land-labor ratio and that it *declines* in value as one moves to the right along the axis. The origin of co-ordinates is 3, 0 rather than the customary 0, 0. Furthermore, there is another important difference in the measurement scale on the horizontal axis. Normally, a unit distance measures a unit change in the variable. Thus the distance from one to two is the same as the distance from seven to eight. In this graph, however, unit distance along the axis measures a unit change in the

denominator of a fraction whose numerator remains constant. Thus the distance from 3/1 to 3/2 is the same as the distance from 3/7 to 3/8.

First observe that the total, average, and marginal product curves for labor (drawn in solid lines) are exactly the same as in Figures 6.2.1 and 6.2.2. Stage I for labor ends at the point where the average product of labor attains its maximum value. Similarly, stage III for labor begins at the point where the marginal product of labor is zero (total product

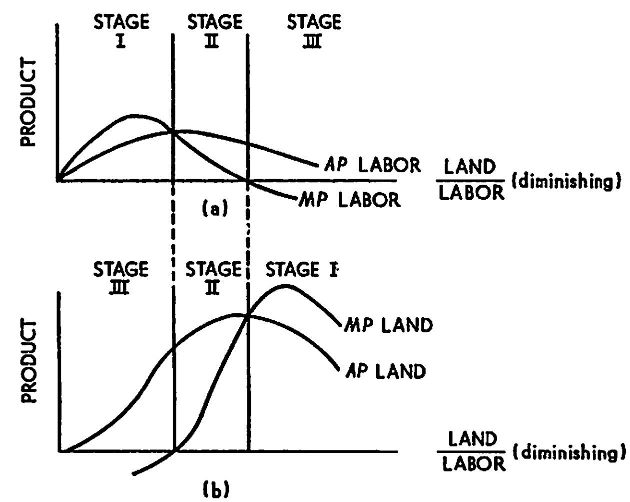

FIGURE 6.3.6

SYMMETRY OF THE STAGES OF PRODUCTION

is a maximum). For greater clarity this is also illustrated in panel *a*, Figure 6.3.6, which represents a more general, continuous case.[8]

The dashed lines in Figure 6.3.5 show the average and marginal products of land (the total product of land is the same as the average product of labor). The graph clearly illustrates—and Figure 6.3.6 re-emphasizes—the symmetry of the three stages of production. The marginal product of land is first negative and reaches zero when the total product of land (the average product of labor) is a maximum. Hence large land-labor ratios define stage III, the area in which the

[8] In the discrete case shown in Figure 6.3.5, stage I ends when average labor product attains its maximum value of thirteen. At this point, due to discontinuous data, marginal product exceeds average product—but it is equal at the next input point, where marginal product equals maximum average product equals thirteen. In the continuous case shown in Figure 6.3.6, average product has a *unique* maximum associated with a *unique* land-labor ratio. In this situation, as in Figure 6.3.4, stage I ends where the marginal and average products of labor are equal.

marginal product of land is negative. Similarly, the average product of land attains its maximum at the same land-labor ratio (three-sevenths) as that which reduces the marginal product of labor to zero. Hence this point defines stage I for land and stage III for labor.

Stage II is the same from either point of view—and it is the only stage in which production will take place. From the standpoint of the variable input, stage II lies between the extensive and the intensive margins. But what is extensive for the variable input is intensive for the fixed input, and vice versa. Thus stage II, from the standpoint of the fixed input, lies between the intensive and extensive margins.

The many points discussed in this subsection may be summarized as follows.

Relationships: (1) There is a symmetry of the three stages of production for the variable and for the fixed input. In particular, stage I (or III) for the variable input and stage III (or I) for the fixed input cover precisely the same range of values of the fixed-to-variable-input ratio. Stage II is the same for both. (2) Over the range of rising average returns (product) to the variable input, the marginal returns of the fixed input are negative. Similarly, the range of rising average product of the fixed input corresponds to the range of negative marginal product of the variable input. (3) Stated differently, the extensive margin with respect to the variable input corresponds to the intensive margin for the fixed input, and vice versa. (4) Over the range of input ratio values for which the average product of the variable input is rising, production will not occur because the fixed input is present in uneconomically large proportion—it is used beyond its intensive margin. Over the range for which the marginal product of the variable input is negative, production will not occur because the variable input is present in uneconomically large proportion—it is used beyond its intensive margin. Thus production will occur only in stage II, the range of input ratio values lying between the intensive margins of the fixed and variable inputs, respectively.

6.4 CONCLUSION

The principles of physical production have now been analyzed for a simple case: one variable and one fixed input. But the discussion in subsection 6.3.e brings us close to a more realistic case: production in situations requiring more than one variable input. The illustration could easily be reconstructed to allow both land and labor to be variable inputs, while (say) x pounds of fertilizer per acre is the fixed input. The next chapter is concerned with the theory of production in the multi-input case.

PROBLEM

Below are hypothetical data for a manufacturer possessing a fixed plant who produces a commodity that requires only one variable input. Total product is given. Compute and graph the average and marginal product curves. Save your basic calculations because they form the basis for a subsequent problem in Chapter 8.

Units of Variable Input	*Total Product*	*Average Product*	*Marginal Product*
1	100		
2	250		
3	410		
4	560		
5	700		
6	830		
7	945		
8	1050		
9	1146		
10	1234		
11	1314		
12	1384		
13	1444		
14	1494		
15	1534		
16	1564		
17	1584		
18	1594		

After completing the table and graph, answer the following questions:

1. When marginal product is increasing, what is happening to average product?
2. Does average product begin to fall as soon as marginal product does? That is, which occurs first, the point of diminishing marginal or average returns?
3. When average product is at its maximum, is marginal product less than, equal to, or greater than average product?
4. Does total product increase at a decreasing rate: (a) When average product is rising? (b) When marginal product is rising? (c) When average product begins to fall? (d) When marginal product passes its maximum value?
5. When average product equals zero, what is total product?
6. (a) If average product is to the left of its maximum, which type of input is present in too large proportion? (b) Which is in too small proportion? (c) What are two ways of changing the proportion so as to increase average product? (d) If either of these were done, what would happen to total product?

(e) In view of these facts, is it desirable or undesirable, from a social point of view, for a producer to operate with average product below its maximum (to the left)? With marginal product at its maximum? Why is your answer true in each case?

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Chapter 7 PRODUCTION AND OPTIMAL INPUT PROPORTIONS: TWO VARIABLE INPUTS

7.1 INTRODUCTION

The fundamental physical relationships of production were discussed in Chapter 6 under the assumption that there is only one variable input. The analysis is continued in this chapter for a more general case. Graphically, production is studied under the assumption that there are two variable inputs. One may regard these inputs either as cooperating with one or more fixed inputs or as the only two inputs. The latter situation, of course, is relevant only for the long run. In either case, however, the results of the two-input model are easily extended to cover multiple inputs.

7.1.a—Production Table

The land-labor example used in Chapter 6 may be expanded to introduce the theory of production with two variable inputs. In the illustration we considered an agricultural experiment in which three-acre tracts of land comprised the fixed input. Labor was the variable input, and we obtained eight sample observations corresponding to the cultivation of the three-acre tracts by one worker, two workers, and so on. In the present example the agricultural experiment is pushed further so as to obtain sixty-four sample observations. Land is, in a sense, still the fixed input; but now we suppose there are eight one-acre tracts, eight two-acre tracts, and so on up to eight eight-acre tracts. Each of the sets of eight constant-acre tracts is cultivated by one worker, two workers, etc., up to eight workers. Thus we have samples ranging from one worker on one acre to eight workers on eight acres. The hypothetical data are listed in Table 7.1.1.

The entries in the row corresponding to three-acre tracts of land are exactly the same as the entries in Table 6.2.2. Indeed, in every respect this table is just a "larger" example of the hypothetical experiment in Chapter 6.

In the spirit of Chapter 6, consider land as the fixed input. The entries in each row show the total outputs produced on the stipulated